Problem 1

A medical laboratory makes blood tests to detect some specific illness. If a patient actually has the illness, the test result is with 95% probability positive. If the test is applied to a person not having the illness, it gives with 1% probability a positive result (so-called "false positives"). The overall fraction of people having the illness is 0.5%. A person is tested positively. What is the probability that this person actually has the illness?

Basically, we have to distinguish two cases: an ill person (event B_1) and a not infected person (event B_2). The positive test result will be denoted A.

$$\begin{aligned}
\Omega &= \{(i,t) : I \in \{ill, healthy\}, T \in \{positive, negative\}\} \\
A &= \{(i,t) \in \Omega : t = \{positive\}\} \\
B_1 &= \{(i,t) \in \Omega : i = \{ill\}\} \\
B_2 &= \{(i,t) \in \Omega : i = \{healthy\}\}
\end{aligned}$$

The probabilities are as follows:

$$\begin{aligned}
p(B_1) &= 0.005 \\
p(B_2) &= 1 - p(B_1) \\
&= 0.995 \\
p(A|B_1) &= 0.95 \\
p(A|B_2) &= 0.01
\end{aligned}$$

The overall probability of being tested positively is:

$$\begin{aligned}
p(A) &= p(B_1) \cdot p(A|B_1) + p(B_2) \cdot p(A|B_2) \\
&= 0.00475 + 0.00995 \\
&= 0.0147
\end{aligned}$$

Now we are able to determine the probability asked for by applying Bayes' Theorem:

$$\begin{aligned}
p(B_1|A) &= \frac{p(B_1) \cdot p(A|B_1)}{p(A)} \\
&= \frac{0.005 \cdot 0.95}{0.0147} \\
&\approx 0.323
\end{aligned}$$

There is a probability of about 32.3% that the person is actually ill.

Problem 2

Be X a random variable with exponential distribution. The exponential distribution has the density function:

$$f(x) = \lambda \cdot e^{-\lambda \cdot x}$$

for $x \geq 0$ and with $\lambda > 0$ being a parameter.

1. *Compute $E[X]$ directly.*
2. *Determine the moment generating function $M_X(z)$ and use this to find $E[X]$ and $Var[X]$.*

I apply the formula $E[X] = \int_0^\infty x \cdot f(x)\, dx$ and the rule $\int x \cdot e^{ax} dx = \frac{e^{ax}}{a^2} \cdot (ax - 1)$

$$\begin{aligned}
\int_0^\infty x \cdot \lambda \cdot e^{-\lambda \cdot x}\, dx &= \lambda \cdot \int_0^\infty x \cdot e^{-\lambda \cdot x}\, dx \\
&= \lambda \cdot \left[\frac{e^{-\lambda x}}{\lambda^2} \cdot (-\lambda x - 1) \right]_0^\infty \\
&= -\frac{1}{\lambda} \cdot \left[e^{-\lambda x} \cdot (\lambda x + 1) \right]_0^\infty \\
&= -\frac{1}{\lambda} \cdot (0 - 1) \\
&= \frac{1}{\lambda}
\end{aligned}$$

In general, the moment-generating function is computed by using the generalized property $M_x(\theta) = E\left[e^{\theta x}\right] = \int_0^\infty e^{\theta x} \cdot f(x)\, dx$:

$$M_x(\theta) = \int_0^\infty e^{\theta x} \cdot \lambda \cdot e^{-\lambda \cdot x}\, dx$$
$$= \lambda \cdot \int_0^\infty e^{x \cdot (\theta - \lambda)}\, dx$$

If the exponent is at negative infinity then the *e*-function evaluates to zero. A small change to the exponent exploits this mathematical property:

$$M_x(\theta) = \lambda \cdot \int_0^\infty e^{-x \cdot (\lambda - \theta)}\, dx$$
$$= \lambda \cdot \left(0 - \left(-\frac{1}{\lambda - \theta}\right)\right)$$
$$= \frac{\lambda}{\lambda - \theta}$$

Because of $E\left[X^n\right] = M_x^{(n)}(0)$ and $n = 1$:

$$\mu = E\left[X^1\right] = M_x^{'}(0)$$
$$= \frac{\lambda}{\lambda - \theta} \frac{d}{d\theta}$$

By deriving, we get:

$$f = \lambda$$
$$f' = 0$$
$$g = \lambda - \theta$$
$$g' = -1$$
$$M_x^{'}(\theta) = \frac{f'g - fg'}{g^2}$$
$$= \frac{\lambda}{(\lambda - \theta)^2}$$
$$M_x^{'}(0) = \frac{\lambda}{(\lambda - 0)^2}$$
$$\mu = E[X] = \frac{1}{\lambda}$$

This result is – as expected – equal to the one we obtained using the direct computation of $E[X]$.

The variance is closely related to the expectation:

$$\begin{aligned}\sigma^2 &= Var[X] = E\left[X^2\right] - E[X]^2 \\ &= M_x^{''}(0) - M_x^{'}(0)^2\end{aligned}$$

The latter part of the formula is known at this point but we still need to determine $M_x^{''}(\theta)$:

$$\begin{aligned}M_x^{''}(\theta) = M_x^{'}(\theta)\,\frac{d}{d\theta} \\ &= \frac{\lambda}{(\lambda-\theta)^2}\,\frac{d}{d\theta} \\ &= \lambda\cdot(\lambda-\theta)^{-2}\,\frac{d}{d\theta} \\ &= 2\lambda\cdot(\lambda-\theta)^{-3}\end{aligned}$$

Replacing $\theta = 0$ leads us to:

$$\begin{aligned}M_x^{''}(0) &= 2\lambda\cdot(\lambda-0)^{-3} \\ &= \frac{2}{\lambda^2}\end{aligned}$$

Finally, the variance is:

$$\begin{aligned}\sigma^2 &= M_x^{''}(0) - M_x^{'}(0)^2 \\ &= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 \\ &= \frac{1}{\lambda^2}\end{aligned}$$

Problem 3

Be X a random variable with gamma distribution. The gamma distribution has the density function:

$$f(x) = \frac{\lambda \cdot (\lambda x)^{\alpha-1} \cdot e^{-\lambda x}}{\Gamma(\alpha)}$$

for $x \geq 0$ with $\lambda > 0$ and $\alpha > 0$ being parameters, and $\Gamma(x)$ being the well-known Gamma function:

$$\Gamma(t) = \int_0^\infty x^{t-1} \cdot e^{-x}\, dx$$

(which is like the factorial function for real numbers: for $n \in N$ we have $n! = \Gamma(n+1)$).

1. *Compute the moment generating function $M_X(z)$.*
2. *For α being an integer, compare the resulting $M_X(z)$ with that of the exponential distribution and give an interpretation.*

To obtain the moment generating function we use the formula $M_x(\theta) = E\left[e^{\theta x}\right] = \int_0^\infty e^{\theta x} \cdot f(x)\, dx$. Therefore, we get:

$$\begin{aligned} M_x(\theta) &= \int_0^\infty e^{\theta x} \cdot \frac{\lambda \cdot (\lambda x)^{\alpha-1} \cdot e^{-\lambda x}}{\Gamma(\alpha)}\, dx \\ &= \int_0^\infty \frac{\lambda^\alpha \cdot x^{\alpha-1} \cdot e^{x \cdot (\theta-\lambda)}}{\Gamma(\alpha)}\, dx \end{aligned}$$

Now we substitute:

$$\begin{aligned} y &= -x \cdot (\theta - \lambda) \\ x &= -\frac{y}{\theta - \lambda} \\ dy &= -(\theta - \lambda)\, dx \end{aligned}$$

So we get:

$$\begin{aligned} M_x(\theta) &= \int_0^\infty \frac{\lambda^\alpha \cdot \left(\frac{y}{\theta-\lambda}\right)^{\alpha-1} \cdot e^{-\frac{y}{\theta-\lambda}\cdot(\theta-\lambda)}}{\Gamma(\alpha)} \left(-(\theta-\lambda)\right) dy \\ &= \lambda^\alpha \cdot \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{1}{\theta-\lambda}\right)^\alpha \cdot \int_0^\infty y^{\alpha-1} \cdot e^{-y}\, dy \end{aligned}$$

Let us take a closer look at the Gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx$$

Surprisingly, the integral on the right side of the moment generating function is the Gamma function. Thus $M_X(z)$ can be rewritten:

$$M_x(\theta) = \lambda^\alpha \cdot \left(\frac{1}{\theta-\lambda}\right)^\alpha$$
$$= \left(\frac{\lambda}{\theta-\lambda}\right)^\alpha$$

If α is an integer the distribution is known as Erlang distribution. For $\alpha = 1$ the moment generating function of the exponential distribution and the Gamma distribution are the same, you might call the exponential distribution a special case of the Gamma distribution.

We can show that for $\alpha = 1$ this relationships always hold true:

$$f(x) = \frac{\lambda \cdot (\lambda x)^{\alpha-1} \cdot e^{-\lambda x}}{\Gamma(\alpha)}$$
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx$$
$$f(x) = \frac{\lambda \cdot (\lambda x)^{\alpha-1} \cdot e^{-\lambda x}}{\int_0^\infty x^{\alpha-1} \cdot e^{-x}\, dx}$$
$$f_{\alpha=1}(x) = \frac{\lambda \cdot (\lambda x)^{1-1} \cdot e^{-\lambda x}}{\int_0^\infty x^{1-1} \cdot e^{-x}\, dx}$$
$$= \frac{\lambda \cdot e^{-\lambda x}}{\int_0^\infty e^{-x}\, dx}$$
$$= \lambda \cdot e^{-\lambda x}$$

Problem 4

Your software company has developed a CAD program. Due to various reasons the development team came up with two different user interfaces for the program, and the company wants to pick one for delivery to the customers. Your boss asks you to design a study giving a "good" answer to this question (with "good" meaning nothing specific, just "our customers will like us"). Make a proposal for the first steps of this study.

This is an "open" problem, there is no "right" or "wrong" solution; use your imagination. You should at least:

- *clearly define the system under study*
- *define some performance measures (as a starting point: consider what a CAD program is actually used for)*
- *identify important input variables (technical ones like the expected user equipment [capabilities of graphics card, monitor etc.] as well as qualitative ones [how much knowledge about computers do users already have, etc.])*
- *decide on the factors and their levels*
- *propose measurement or simulation setups studying these measures*

A CAD (computer-aided design) program is typically used to design, modify and analyze complex machines or buildings. This branch of software developed in the early days of the computers (the first computer built by Zuse was actually used to evaluate tables related to the construction of buildings) and still changes its face up to now.

Basically, there is only one relevant factor: the time needed by a user to fulfill a special task. Unfortunately, one cannot easily define "Who is a user ?" and "What is his/her special task ?" and furthermore it seems to be impossible to get an exact impression of the verbs "design, modify and analyze".

Users are often divided into three clusters: beginner, advanced and professional users. Each cluster needs to be handled separately because of different qualification and requirements. Not enough to take their mental properties into account, one will have to take a look at their financial background and their technical equipment. A professional user may be willing to invest a quite huge amount of money, buy a fast computer and acquire proprietary techniques because he/she uses the software for a long period and earns his/her living by working with the software. On the other hand, a beginner often just needs to fulfill a single task without spending much time reading the user's guide. Microsoft tends to offer wizards for each and every user's task – something that is very helpful to beginners.

Nowadays the GUIs take care of these user clusters: they are able to be operated by only using the mouse (or something similar, i.e. a trackball) or by employing the keyboard. On first glance, it may seem old-fashioned to type in the numbers etc. but it proved to be far superior in terms of speed and accuracy. In my eyes, more than 95% of all professional users use the keyboard approach whereas maybe 95% of all beginners prefer the mouse because their common workplace is operated by a mouse (modern GUIs like MacOS, Windows …). A professional told me the main difference: a beginner works in a more creative way; you might say he/she "plays", whereas a professional just wants to put an assigned problem into action.

A beginner usually does not want to upgrade his/her computer system in order to run a CAD program. Factors like speed (in GHz), the screen size (in MPixel), the color depth (in bits) and the hardware support (input devices or accelerating cards like modern 3D graphic adapters) play an important role when an optimal performance needs to be achieved that a professional user is interested in.

It is quite common that the user interface offers far more settings and options than it can display on the screen. Therefore, it has to adapt to the current situation, for example, by tool tips, intelligent toolbars or wizards. The language the software comes with is often underestimated, too.

All these factors can be measured exactly but their true influence on the time needed to fulfill the given task may vary from time to time based on personal preferences. The weighting of these factor follow my own experiences:

very important	important	neutral
– immediate feedback on operations – accepted input devices – CPU speed – amount of RAM available – availability and accessibility of the help system	– language of the GUI (and some localization issues) – screen resolution – customizable menus and / or toolbars – extensibility – support for macros	– hardware acceleration (graphics card) – color depth

Table 1: Factors and their levels

Some years ago, the hardware acceleration was more important than it is today. The main reason that I rate it "neutral" lies in the observation that even low-cost hardware offers a very broad support. All students known to me – who study architecture or construction – never felt the need for a faster graphics card. The only exception is the process of creating a 3D presentation but these cases are very rare and can be neglected (there is specialized 3D software available, anyway).

People often forget to evaluate the "feel-good-value" of the user interface. A nice look empowers the user to work more intensively and to get less frustrated. Unfortunately, it is impossible to find a good scale for that factor. Only statistically supervised end-user tests may lead to a conclusion whether they actually like the design offered by the software.

Indeed, there are some "hard facts". Each user (or at least most users) has to perform many so-called "common tasks" like inserting a predefined module, moving a line or saving a file. These operations may vary only slightly and thus you can actually measure the time an average user needs. Depending on the target audience the company ought to define an operation mix – a weighted mixture of operations usually performed during the process of fulfilling a task. The final sum then has to be minimized. With respect to decent results, one will have to design, modify and analyze a reasonable complex "real world" scenario and repeat it multiple times to remove or suppress inaccuracies in the measurement process.

In my eyes, the weighted combination of the timing results and the impression of the design (sometimes called look-'n'-feel) seems to deliver all the reasoning you need to convince your boss of either the first or the second GUI.